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Self-interacting fundamental strings and black holes

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Abstract

We study the size distribution of very massive close string states and the typical string configuration as one slowly increases the string coupling, both in the case of zero and of non-zero Neveu-Schwarz charges. The computations are performed rigorously in string theory, starting from quantities that are well-defined in the theory and therefore clarify previous works on the subject which were based on various approximation techniques.

We find that, starting from a value of the coupling in agreement with the one predicted by the black hole correspondence principle, the string ensemble is dominated in any dimensions by compact states whose size is within the correspondent black hole horizon radius, which is of the order of the string scale at the black hole/string transition/matching point.

Contents

1 Introduction

Black holes obey a set of laws formally identical to the thermodynamic ones. In particular to every black hole are associated an entropy and a temperature. These have to be accounted for by an underlying microscopical theory, which should provide us not only with an interpretation of black holes entropy, but also with the possibility of deriving the whole set of the thermodynamic laws from first principles.

According to Bekenstein's principle, the black hole entropy is proportional to the area of the horizon (plus corrections[1]); for a quantum statistical

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[1]The more general formula for the entropy of a black hole, including higher derivative

ensemble, instead, the entropy is defined as the logarithm of the number G of microstates:

$$S = \ln(G). \tag{1}$$

Relating the two definitions represents the entropy issue. It is necessary to individuate the correct ensemble of microstates accounting for S: we need therefore a quantum gravity theory. String theory represents probably the best candidate nowadays for such a theory and it has a general principle (known as the "String-Black holes correspondence principle" [2, 3]) individuating those microstates.

A key role in the correspondence principle is played by the relationship between the Schwarzschild radius of black holes and the string length scale, the two characteristic scales set by black hole Physics and string theory. At the matching point these two should be equal [3] and we would expect only string states whose size is within the Schwarzschild radius to be related to the black hole description[2]. We need therefore to be able to study properties of sets of states at given mass, charge, angular momentum but also "size".

In this work we want to focus on two aspects of this problem: 1) on properly computing the entropy of microstates in string theory in terms of their mass, (Neveu-Schwarz) charges and size, 2) on determining how the configuration of string states ensembles (their distribution in terms of size) changes by varying (adiabatically) the string coupling.

This has never been done before rigorously in string theory formalism, because of two principal reasons. On one hand, it is not straightforward, in the quantum theory of strings, to define an operator measuring the (average) size of string states and this complicates the analysis already at tree-level. On the other hand, taking into account string self-interactions requires one-loop string computations which are difficult to be performed. Because of all this, the partition function for single-string states constrained[3] in both mass, charges and size is unknown both at tree-level (free string) and at non-zero string coupling. The main result of this work will therefore be to compute it in a well-defined way and to obtain the entropy from it.

Note that the free-string entropy as a function of mass, charge and size would now receive corrections in both the non-BPS and the BPS cases (renormalization of the size), however we will concentrate on the self-energy contribution for the non-BPS case. In particular, the one-loop corrections to our

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terms, is Wald's one [1].

[2]Consider for example the gravitational binding to mass ratio, or the fact that the metric sourced by the microstates should differ from the one of a black hole only at distance lower than the horizon radius.

[3]We will use the microcanonical ensemble, as explained in the following.

tree-level results will be obtained and discussed elaborating on the results in [4].

A part from the interest in relation with black holes, studying string configuration (size) and entropy will also clarify the link between the string theory and the random-walk pictures in full details[4].

We consider closed string, and perform our analysis both in the bosonic and in the superstring theory (type IIB or IIA). We will deal with states carrying both zero and non-zero charges of the Neveu-Schwarz type (Kaluza-Klein momentum and winding numbers).

The approach we follow goes *beyond* those used so far, which assumed some sort of approximation in the description of strings (random-walk, polymers, thermal scalar, ...) and led sometimes to results actually being in contradiction (see [5, 6]). Indeed, we will compute the partition function from first principles consistently within String Theory.

We begin in Section 2, by reviewing the String-Black Holes Correspondence Principle. Special emphasis is given to the role played by the value of the horizon radius of the black hole and the corresponding requirements for the size of the string microstates. This section establishes the basis and motivation for the present study. We also review the existing literature on the subject and the controversial and obscure points.

3 and 4 are introductory sections where we discuss and specify our statistical ensemble of closed string states and the definition of "size" of a string[5].

The rest of the paper is divided in two parts: the first one (section 5) dealing with the size distribution of highly excited *free* string states, the second one (section 6) focusing on the corrections to the size distribution due to self-interaction of the string.

Finally, we comment and conclude.

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[4] In the past, instead, the only quantities computed within String Theory and compared with the random-walk picture were just the root mean square radius and some density-density correlators, as we will mention.

[5] An important part of the proof that we are indeed computing the correct partition function will be presented later, in section 5.1.3.

2 The String-Black Holes correspondence principle

In this section, for simplicity, we consider the case with zero charges[6]. String theory and black holes' physics set two characteristic length scales:

R_{bh} the black hole horizon radius (Schwarzschild radius)

$l_s = \sqrt{\alpha'}$ the string length scale

so that

$$
\begin{cases}
\text{if} \quad R_{bh} \gg l_s \quad \text{general relativity description is reliable} \\[2ex]
\text{if} \quad R_{bh} \lesssim l_s \quad \text{strings feel space-time as flat, } \alpha'\text{-corrections are important,} \\
\qquad\qquad\qquad\quad \text{string theory description is reliable}
\end{cases}
$$

The "String-Black Holes Correspondence Principle" [2, 3] says that a black hole is described by an ensemble of excited string and/or D-brane states (depending on the type of charges the black hole possesses).

There are two possible interpretations of the Principle:

- a **physical process** (Hawking radiation) where the black hole decreases its mass M, therefore reducing the value of its Schwarzschild radius[7] $R_{bh} \sim (G_N M)^{\frac{1}{d-2}}$ until $R_{bh} \sim l_s$ where a transition to an excited string states takes place. In this case the (closed) string coupling g_s is fixed, while M varies.

- **two complementary descriptions** valid in different regimes at equal mass. In this case g_s varies, while M is fixed. This is the approach that we will use in this work.

The two descriptions can be compared at determined values of the coupling when they are both valid. The possibility of equating the black hole entropy (proportional to a power of its mass) and the string one (proportional to the square root of its mass) relies on the fact that the first is constant in Plank units, the second in string ones and therefore the entropies match at a determined value of the string coupling. There, at the matching point, entropy and mass[8] of the black hole and the string are equal and it is found

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[6]For non-zero charges the Schwarzschild radius (horizon radius) in the string frame and that in the Einstein frame are different and care must be used in stating the correspondence principle, see [3] and footnote 46.

[7]We are in $d = D - 1$ extended spatial dimensions. In perturbative regime, which will be the one we will work in, we relate Newton's constant to the string length as $G_N \sim g_s^2(\alpha')^{\frac{d-1}{2}}$, considering compactified dimensions with curvature radii $r_i \sim \sqrt{\alpha'}$.

[8]And charges when they are present.

that, in units of α' [3],

$$g_s \sim M^{-\frac{1}{2}}, \qquad R_{bh} \sim l_s \tag{2}$$

independently of the number of dimensions. Since we are to consider very massive string states, this value for the string coupling turns out to be sufficiently small to allow perturbation theory.

We would expect that only states whose size is within the black hole horizon radius can be related to the black hole at the transition/matching point. It is therefore interesting and important to find the entropy of string microstates depending on both mass and size.

The main novel result of this work is the definition and the computation in a fully consistent way within string theory of the partition function (and therefore entropy) for states constrained in mass, (charge) and size[9] (in the microcanonical ensemble) and the study of the string configuration when the coupling g_s is slowly increased.

In the past, a few attempts have been made to study such issues: [8], [9] and especially [5], [6] (see also [10]).

In [5], it was employed a thermal scalar formalism, interpreting the size of the bound states of a certain scalar field as the size of the excited *open* string states. The thermal scalar is a formal device capable to give us some statistical information about the string system (string gas). Nevertheless its relation to the string states remains open. In particular, interpreting the size of bound states of the thermal scalar as the size of the string states in Minkowski space is not obvious since, as the authors of [5] remark themselves, the thermal scalar has no dynamical meaning. However we will show how the results in [5], taking into account the differences between open and closed strings, compare to ours.

The polymer and string bit picture for strings was the approximation used in [8], [9]. This is related to the random-walk one in the idea of the string depicted as formed by an ensemble of "bits". But these models have many open issues, in particular for the superstring (see the discussion in [11] and in [8] itself).

The results obtained in [6] merge instead physical intuition with computations in a model of (bosonic open) "strings", believed to be valid in a large number of dimensions ($d \gg 1$), which did not take into account any Virasoro constraint. The results of [6], if correct, are somehow puzzling and raise some concern about the viability of perturbation theory in string theory, as we will now discuss.

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[9]It would also be possible to characterize the states through their angular momentum, obtaining the entropy as a function of mass, charge, size and angular momentum, following [7].

The tree-level logarithm of the string partition function for states at large masses $M = \sqrt{N}$, resulting from the computation in their model (formula (2.28) in [6], using formulas (2.10, 2.27, 2.28)) reads

$$\log(Z) = \sqrt{\frac{\pi^2 d}{6} N} \left(1 - \frac{9}{4}\frac{1}{R_s^2}\right) \tag{3}$$

and is obtained at large mass through an expansion for $\delta \sim \frac{l_s}{R_s^2} \ll 1$, therefore is a result valid for large string sizes[10] R_s.

This result is puzzling, since for large size and masses we would expect the string to behave like a random-walk. Indeed, the authors of [6] need to combine by hand the result from the computation with a corrective factor $e^{-R_s^2 M^{-2}}$ to the partition function in order to obtain physically reasonable results. The problem is that the operator measuring the size of string and the Fock space used in [6] are not well-defined. The same (as for the operator) happens in [10].

For what concerns their estimate of the one-loop corrections (squared average mass-shift) for open self-gravitating strings, let us consider their result from formula (3.26) and the line above (3.25). At small string coupling they find a typical size $R_{av} \sim \sqrt{M}$ for the average string. Therefore the result in [6] for the average squared mass shift is (using the notation there)

$$\delta M^2 = -cg_s^2 M^3 M^{\frac{2-d}{2}} = -cg_s^2 M^2 M^{\frac{4-d}{2}} . \tag{4}$$

We see that for $d = 3$ this result means that at a given[11] g_s, for states with sufficiently large M^2, the correction to the mass squared will be larger than the original value (that is $\delta M^2 > M^2$). This would mean that we cannot apply perturbation theory consistently on the whole string spectrum, in particular not in the limit of large masses.

It is therefore fair to say that the results in [6] need to be verified both at tree and one-loop level[12].

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[10]The result is obtained through an expansion for large R_s, as we said, and not limited in any way by the value of the mass of the states, see formula 2.25 in [6]. A posteriori it is declared that the result is valid only for $R_s^2 < M$ but there is no reason from the computation for doing so.

[11]Recall that g_s in string theory is not a free parameter, but depends on the dilaton vacuum expectation value, although in this work we use it as a useful parameter for investigating the string configuration.

[12]The authors of [6] acknowledge the difficulties and their investigation is on a physically motivated ground. Nevertheless it is important to verify it.

We will perform our calculations rigorously in full-fledged string theory. Our conventions, here and in the following, are

$$\alpha' = 4,\,^{13} \qquad D = d + 1 \text{ large space-time dimensions}$$
$$g_s/g_o = \quad \text{closed/open string coupling.}$$

Furthermore, quantities with a "c" subscript will refer to closed strings, whereas those with an "o" subscript will indicate open strings. Finally, a tilde or a subscript R refer to the right-moving sector of the close string, a subscript L or no tilde refer to the left-moving one.

The computation of the entropy of strings is ultimately connected also with the Hagedorn transition in string theory, but we will deal with single-string entropy and therefore our results do not apply directly.

3 The microcanonical ensemble

We want to determine statistical properties of massive string states, in particular concerning their spatial distribution. We will use the *microcanonical* ensemble[14]. Let us discuss for a moment the definition of microcanonical ensembles on more general ground before specifying our case.

Ensembles are defined by density matrices: in usual statistical mechanics the microcanonical one has the form[15]

$$\rho_E = a_E \delta(E - \hat{H}) \tag{5}$$

where $\hat{H}$ is the Hamiltonian[16] of the system and a_E ensures the normalization of the density matrix

$$\text{tr}[\rho_E] = 1 \tag{6}$$

when traced over the states.

We can try to modify the traditional microcanonical ensemble, fixing the value of other observables, in order to investigate different statistical properties of the system. Considering a discrete observable with associated operator $\hat{Q}$, we can define the density matrix

$$\rho_{E,Q} = a_{E,Q} \delta(E - \hat{H}) \delta(Q - \hat{Q}). \tag{7}$$

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[13]Our results will be written in string units.

[14]We will discuss our preference for the microcanonical ensemble over the canonical one in section 5.1.3, footnote 33.

[15] The expressions for the density matrix are meaningful when applied to the states of a system; with that understanding our notation with Dirac's delta functions is clear.

[16]From now on a ˆ will distinguish an operator from its value(s). At this moment we consider discrete Hamiltonian spectrum.

If Q represents an observable with continuous spectrum, we need to specify a small interval δQ (uncertainty) around the value of the observable we are interested in[17], and define

$$
\begin{aligned}
\rho_{E,Q,\delta Q} &= a_{E,Q,\delta Q}\, \delta(E - \hat{H}) \left(\theta(Q + \delta Q - \hat{Q}) - \theta(Q - \hat{Q})\right) \\
&= a_{E,Q,\delta Q}\, \delta(E - \hat{H}) \int_Q^{Q+\delta Q} \delta(Q - \hat{Q}).
\end{aligned}
\tag{8}
$$

We will let $\delta Q \to 0$, so that we can write

$$
\rho_{E,Q} = a_{E,Q}\, \delta(E - \hat{H})\delta(Q - \hat{Q}).
\tag{9}
$$

The partition function

$$
\begin{aligned}
G(E,Q) &= \operatorname{tr}[\delta(E - \hat{H})\delta(Q - \hat{Q})] \\
&= \sum_\phi \langle\phi|\delta(E - \hat{H})\delta(Q - \hat{Q})|\phi\rangle
\end{aligned}
\tag{10}
$$

gives the number of states having the values E, Q for the chosen observables[18]. It is, therefore:

$$
a_{E,Q} = G(E,Q)^{-1}
\tag{11}
$$

In our case we are interested in the partition function for a certain microcanonical ensemble, defined by fixing the values

- of the (squared) *mass* [19] $M_0^2 = N_{R(L)} + Q_{R(L)}^2$

- of the (squared) *size* R^2 of the strings[20]

- of the Neveu-Schwarz charges $Q_{L,R}$ (see (80)).

Then, the partition function can be written in the form:

$$
G_c(N, R, Q_{L,R}) = \operatorname{tr}[\delta(N_R - \hat{N}_R)\, \delta(N_L - \hat{N}_L)\, \delta(R - \hat{R}_s)\, \delta(Q_L - \hat{Q}_L)\, \delta(Q_R - \hat{Q}_R)]
\tag{12}
$$

Note that, since the squared size is more easily defined in string theory (see section 4), we always define:

$$
\delta(R - \hat{R}_s) \equiv 2R\, \delta(R^2 - \widehat{R^2}_s).
\tag{13}
$$

---

[17]This applies of course also to the energy, if that is the case.

[18]The number of microstates i having values E for the energy, and $Q < Q_i < Q + \delta Q$ in the case of continuous observables.

[19]We define $\hat{N}_L \equiv \sum_{n=1}^\infty \hat{\alpha}_{-n}\hat{\alpha}_n - 1$ as the left-moving level number operator, whose value is fixed once we fix the mass and charges. The right-moving one is similarly defined in terms of tilded oscillators. Note that here we write the tree-level mass.

[20]As we will see in section 4, the squared size is more easily defined in string theory.

We choose not to write formal definitions of partitions functions in terms of $\delta(R^2 - \widehat{R^2}_s)$ in order to make more readable our formulas. It is understood, then, that in the following we always actually deal with the operator $\widehat{R^2}_s$, and use (13) in the formal definition of our partition function.

The apparently straightforward computation of (12) has its main difficulty in defining the string size operator $\widehat{R^2}_s$ in full consistency with the quantization of the theory. We will show in section 5 how in fact (12) can be actually computed within String Theory, but first we will discuss in more detail the issue of defining $\widehat{R^2}_s$.

4   Measuring the size of strings

The size of a string is usually covariantly defined in the classical theory, as the average (squared) size through the formula

$$
\mathcal{R}_{\text{cl}}^2 = \frac{1}{\Delta\sigma_+\Delta\sigma_-} \int_0^{\Delta\sigma_+} \int_0^{\Delta\sigma_-} (X_\perp(\sigma_+, \sigma_-))^2 \quad \sigma_\pm = \sigma \pm \tau, \tag{14}
$$

where

$$
X_\perp^\mu(\sigma_+, \sigma_-) \equiv \tilde{X}^\mu - p^\mu \frac{p \cdot \tilde{X}}{p^2} \tag{15}
$$

and

$$
\tilde{X}^\mu \equiv X^\mu - X_{\text{cm}}^\mu \tag{16}
$$

where X_{cm}^μ is the center of mass motion of the string and p^μ is its center of mass momentum. In this way $X_\perp^\mu$ represents the projection of the oscillator part $\tilde{X}^\mu$ of the string coordinate orthogonally to the center of mass momentum of the string. $\Delta\sigma_\pm$ are the periodicities of $\sigma_\pm$.

This classical quantity is the one that has been used in the literature when comparing strings to black holes, regarding size (for example see [6, 10, 12]) and this one will be dealt with in this work.

Using the expansion

$$
\tilde{X}^\mu(\sigma, \tau) = \sqrt{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left(\frac{\alpha_m^\mu}{m} e^{i(\tau-\sigma)} + \frac{\widetilde{\alpha}_m^\mu}{m} e^{i(\tau+\sigma)} \right), \tag{17}
$$

we find, in the string rest frame,

$$
\mathcal{R}_{\text{cl}}^2 = \mathcal{R}_L^2 + \mathcal{R}_R^2, \tag{18}
$$

9

where

$$\mathcal{R}_L^2 \equiv 2 \sum_{i=1}^{d} \sum_{n=1}^{\infty} \left(\frac{\alpha_{-m}^i \alpha_m^i + \alpha_{-m}^i \alpha_m^i}{n^2} \right) . \tag{19}$$

$\mathcal{R}_R^2$ is defined as $\mathcal{R}_L^2$ with the correspondent tilded quantities.

The seemingly most straightforward choice, at this point, would be to keep the definition (14, 18, 19) also when quantizing the string and promoting the α_m^μ's (and $\widetilde{\alpha}_m^\mu$'s) to operators $\hat{\alpha}_m^\mu$, $\hat{\widetilde{\alpha}}_m^\mu$. Unfortunately, three evident issues regarding such operator are:

- its definition is gauge-dependent,

- the operator has a zero-order contribution proportional to

$$\sum_{n=1}^{\infty} \frac{1}{n} \tag{20}$$

  which needs to be interpreted and regularized (see [13]),

- the insertion of this operator in a path-integral is problematic because $\widehat{\mathcal{R}^2}$ defined in this way is not BRST invariant, in a covariant gauge formulation of String theory, or equivalently, in an Old Covariant quantization scheme, it generates unphysical states when applied to physical ones.

  Instead, when using a Light-Cone gauge quantization, the presence in $\widehat{\mathcal{R}^2}$ of the longitudinal oscillators $\alpha_m^- = (p^+)^{-1} L_m^{\text{transverse}}$ leads to a complicated interacting theory for the transverse oscillators and poses ordering problems when used in composite operators, such as our desired delta function.

It appears therefore clear that one cannot carry over consistently in String Theory the minimal program of a) considering formula (12), b) expressing the delta function[21] in terms of operators and using the $\widehat{\mathcal{R}^2}$ defined in the naive way we just illustrated, c) tracing over physical string states.

We will instead work only with quantities[22] that are well-defined in string theory. From them, we will define and compute the partition function in terms of mass, charge and size. The classical value (14) must obviously be recovered in the (semi)classical limit. We will use this as a key test in verifying the correctness of our definition for the operator in question (see section 5.1.3).

The procedure we will follow to obtain (12) will then be:

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[21]For example through Fourier transform or similar.
[22]Namely amplitudes.

a) compute the partition function[23]

$$G^* = \text{tr}[\delta(N_R - \hat{N}_R)\,\delta(N_L - \hat{N}_L)\,\delta(r - \hat{R})\,\delta(Q_L - \hat{Q}_L)\,\delta(Q_R - \hat{Q}_R)] \quad (22)$$

for an operator size $\hat{R}$, to be defined starting from well-defined string amplitudes and from a physical procedure to measure the size of objects (see section 5).

b) prove that the operator $\widehat{R^2}$ defined by this procedure recovers the classical value and form given by (14, 18, 19), when evaluated on the string ensemble, and that therefore it is the correct string squared size operator $\widehat{R^2}_s$, so that G^* is indeed $G_c(N, R, Q_{L,R})$ as written in (12).

Let us now define and obtain G^*.

5 Size distribution of highly excited free string states

To avoid cluttering of formulas, for the moment we will consider states with zero charges; in section 5.2 we will extend our method to non-zero fixed charges.

Let us start by considering the on-shell string amplitude defined as[24]

$$A_{\text{closed}} = \frac{g_s^2}{G_c(N)} \int d^2 z \sum_{\phi_{|N}} \langle \phi | V(k', 1) V(k, z) | \phi \rangle. \quad (23)$$

The sum is over physical string states at fixed squared mass N extended in the large dimensions. The on-shell vertex operators[25]

$$V(k, z) = \frac{2}{\alpha'} e^{ik \cdot X} (\partial X^\mu - \frac{i}{2}\psi^\mu k \cdot \psi)(\bar{\partial} X^\nu - \frac{i}{2}\tilde{\psi}^\nu k \cdot \tilde{\psi})\,\xi_{\mu\nu}, \quad k^2 = k^\mu \xi_{\mu\nu} = 0 \quad (24)$$

---

[23]Here we again have

$$\delta(r - \hat{R}) \equiv 2r\,\delta(r^2 - \widehat{R^2}), \quad (21)$$

as in (13).

[24]$G_c(N)$ is the number of closed string states at mass level N. Note that $G_c(N) = G_o(N)^2$, where $G_o(N)$ is the number of open string states at level mass N.

[25]Our vertex operators do not have the usual string coupling factor carried by the string vertex operators. This is because for clarity we have decided to explicitly show all string coupling factors in front of our amplitudes in this paper.
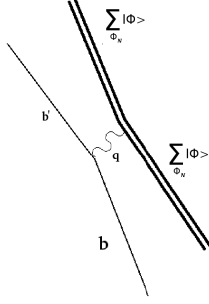
Figure 1: *Scattering process $b + \sum_N \phi \to b' + \sum_N \phi$. At low momentum transfer the process is dominated by the massless channel, which is represented here.*

represent gravitons (actually a superposition of graviton, dilaton and Kalb-Ramond field, in the following indicated by the letter b). Here, X, ψ are, respectively, the space-time string bosonic and fermionic coordinates.

Our goal will be computing (23) and show how from it we can obtain (22).

The amplitude can be conveniently re-written as a trace inserting the density matrices (mass projectors) $\rho_N, \tilde{\rho}_N$, where

$$\rho_N = \frac{1}{\sqrt{G_c(N)}} \delta(N - \hat{N}_L) = \frac{1}{\sqrt{G_c(N)}} \frac{1}{2\pi i} \oint \frac{dw}{w^{N+1}} w^{\hat{N}_L} . \qquad (25)$$

Note that $N_R = N_L = N$ for the case we are considering.

We obtain

$$A_{\text{closed}} = g_s^2 \int d^2 z \, tr[V(k', 1)V(k, z)\rho_N \tilde{\rho}_N]. \qquad (26)$$

In the form (26), the amplitude can be computed as a one-loop two point amplitude for the probes b (projected on "initial" states of mass-level N and without integrating over the zero modes). This ensures that only physical states enter in the trace and therefore that this on-shell amplitude is well-defined in string theory.

The process accounted for by (23) is the one depicted in figure 1, that is the scattering of b off the average string state at mass-level N. Note that what we are computing here is an amplitude, not an *inclusive* cross section, as had instead been done in [14] for the bosonic string and later in [15] for the superstring.

We will study this amplitude for elastic scattering (when $q^2 = \vec{q}^{\,2}$), low momentum transfer $(k + k')^2 = q^2 = \vec{q}^{\,2} \to 0$ and energetic probes and

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massive states[26] such that $-k \cdot p \gg 1$ (see section 5.1.2). In this limit the amplitude can be written as (proof will be given in section 5.1.2)

$$A_{\text{closed}} \sim \frac{(E\sqrt{N})^2}{q^2} F_b(q^2) F_N(q^2) \tag{27}$$

and $F_i(q^2)$ is the *form factor* of the state i. In our formula we used the symbol F_N because we are interested in a microcanonical ensemble at fixed mass level number $N = M^2$.

We also know that the form factor is the Fourier transform of certain spatial distributions of the string. For example, if the probe b is a gauge field excitation then we recover the relative charge spatial distribution $\mu_Q(\vec{r})$, if b is a graviton (as in our case), we recover the mass spatial distribution $\mu_N(\vec{r})$.

We therefore write

$$F_N(q^2) = \int dq\, e^{i\vec{q}\cdot\vec{x}} \mu_N(\vec{r}) \tag{28}$$

and we recall that (we consider here the case of spherical symmetry, which will occur in our case, $\int d\Omega_d$ is the angular integral)

$$\begin{aligned}
\int d\Omega_d\, \mu_N(\vec{r}) &= \Omega_d\, r^{-(d-1)} \mu_N(r) \tag{29} \\
&= r^{-(d-1)}\, G_c^{-1}(N)\, tr[\delta(r - \hat{R})\delta(N - \hat{N}_R)\, \delta(N - \hat{N}_L)]
\end{aligned}$$

using (25, 26) and the definition of mass spatial distribution in a quantum theory for states with mass $M = \sqrt{N}$.

At this point we define

$$\begin{aligned}
G^* &= tr[\delta(N - \hat{N}_R)\, \delta(N - \hat{N}_L)\, \delta(r - \hat{R})] \\
&= G_c(N)\Omega_d\, \mu_N(r)\,, \tag{30}
\end{aligned}$$

where we have used (28, 29).

The operator $\widehat{R^2}$ has been therefore defined in a operational way. What we are doing here really represents the correct method for physically defining and measuring the size of an object: through scattering processes. The average value of $\widehat{R^2}$ is given by

$$\langle \widehat{R^2} \rangle = -2d\, \partial_{q^2} F_N(q^2)|_{q^2=0}\,. \tag{31}$$

In order to show that G^* is indeed the looked-for partition function (12), we have to demonstrate that the operator $\widehat{R^2}$ defined in this way really measures the size of a string, that is to say, it recovers the correct classical value

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[26]This corresponds to the Regge limit of our amplitude.

and form (14, 18, 19) in the (semi)classical limit. This can be done using (31), which enables us to compare the average value of our operator $\widehat{R^2}$ with the average size of the string computed classically with (14). We will prove this in section 5.1.3[27].

An additional new result that we obtain here is the computation of the form factor for a string ensemble at fixed squared mass. Note that this has not been previously done in the literature[28].

In the following, we compute (23) both for the bosonic string theory and for the superstring. The computations are in fact very similar. We therefore discuss first and at length the superstring, and later address the bosonic.

5.1 States with no charge

As we said, we concentrate at first on the case where the string states in our ensemble carry no charge. The formulas are in fact neater and all the steps can be discussed in a clearer way. In section 5.2, instead, we will consider non-zero charges of the Neveu-Schwarz type.

5.1.1 The String Spatial Distribution

The superstring.

In computing (23, 26), we will make use of the relation[29] [18]

$$A_c(1234; \alpha', g_s) = \frac{\pi i g_s^2 \alpha'}{g_o^4} \sin(\pi \alpha' t) A_o(s, t; \frac{\alpha'}{4}, g_o) \tilde{A}_o(t, u; \frac{\alpha'}{4}, g_o). \tag{32}$$

where s, t, u are Mandelstam variables.

The amplitude that we will compute is therefore[30]

$$A_o(s, t; 1, g_o) = g_o^2 \int dy \, \mathrm{tr}[V_{\mathrm{open}}(k', 1) V_{\mathrm{open}}(k, y) \rho_N] \tag{33}$$

---

[27]Proving this correspondence will actually require that we formally use the canonical ensemble, but that will be straightforward to obtain from our formulas, as we will show.

[28] Our formulas, and therefore the result that we will obtain, are indeed very different from the ones in [16]. In our case the interpretation of $F_N(q^2)$ as a form factor is justified, according to scattering theory, whereas in [16] it could not be accepted because the form factor of an object cannot be obtained from the *semi-inclusive* cross-section of its decay. In fact this was acknowledged by the author of [16] himself in the successive paper [17] (see section 3 there), where a different and more correct interpretation of the results was proposed.

[29]Here we have explicitly reinstate α' in order to present clearly the formula. Remember that in the computations we will always set $\alpha' = 4$.

[30]This well-defined string amplitude can be computed both in the covariant and in the light-cone gauge, at each one's own convenience. We present here the results, which are independent of the chosen gauge.

with, given (24),

$$V_{\text{open}}(k, y) = \frac{e^{ik \cdot X(y)}}{\sqrt{2\alpha'}} \left(iy\xi \cdot \partial_y X(y) + 2\alpha' k \cdot \psi(y)\xi \cdot \psi(y) \right) \qquad k^2 = k \cdot \xi = 0 \,. \tag{34}$$

We will consider the limits $t \equiv -q^2 = -(k + k')^2 \to 0$, $-k^0\sqrt{N} \gg 1$. The leading term of the amplitude in this limit can be calculated using the OPE (see [16])

$$V_{\text{open}}(k', 1)V_{\text{open}}(k, y) \underset{y \to 1}{\sim} g_o^2 2\xi \cdot \xi'(1 - q^2)(1 - y)^{2k' \cdot k - 2} y^{2k \cdot \hat{p}} e^{iq \cdot \hat{X}_O(1)} e^{iq \cdot \hat{x}} \tag{35}$$

where $\hat{X}_O$ indicates the oscillator part of X. Note that, by performing this OPE, we are actually using the property of factorization of (string) amplitudes. This will turn out to be very useful: factorizing two external legs of an amplitude, the momentum square q^2 flowing along the connecting propagator is a continuous variable, allowing analytic continuation.

The amplitude can be written as

$$A_o = g_o^2 A_o^{\text{zero modes}} A_o^{\text{oscillators}} \tag{36}$$

By writing $y = e^{-\epsilon}$ with $\epsilon \to 0$, we find the result[31]

$$A_o^{\text{zero modes}} = -\int d\epsilon \, \epsilon^{q^2 - 2} e^{-\epsilon(2k \cdot p + 1)}(1 - q^2) \tag{37}$$

$$\underset{q^2 \to 0}{\sim} \frac{(2\sqrt{N}E)^2}{q^2} \sqrt{F_b(q^2, E)}(2\sqrt{N})^{-q^2}. \tag{38}$$

where we have defined

$$F_b(q^2, E) \equiv e^{-2q^2 \ln(2E)} \tag{39}$$

and $E \equiv k^0$.

Therefore,

$$A_c(1234; 4, g_s) \sim \pi^2 i g_s^2 \frac{(2\sqrt{N}E)^4}{q^2} F_b(q^2, E)(2\sqrt{N})^{-2q^2} A_o^{\text{oscillators}} \tilde{A}_o^{\text{oscillators}} \tag{40}$$

where

$$A_o^{\text{oscillators}} = \text{tr}[e^{iq \cdot X_O(1)} \rho_N] \tag{41}$$

and we have expanded $\sin(-\pi t) \sim -\pi t \sim \pi q^2$.

---

[31]We need to perform the same analytical continuation as for the Veneziano amplitude, as usual in these representation of the string amplitudes.

According to the results and the discussion in [19], we identify $F_b(q^2, E)$ with the form factor for the probe b[32].

It is now straightforward to read the form factor for the target average state at squared mass N:

$$\begin{aligned}
F_N(q^2) &= (4N)^{-q^2} A_o^{\text{oscillators}} \tilde{A}_o^{\text{oscillators}} \tag{42}\\
&= \frac{4N^{-q^2}}{G_c(N)} \frac{1}{(2\pi i)^2} \oint \frac{dw}{w^{N+1}} \oint \frac{d\tilde{w}}{\tilde{w}^{N+1}} \frac{g(w)g(\tilde{w})}{(f(w)f(\tilde{w}))^{d-1}}\\
&\qquad \times e^{-2q^2 \sum_{n=1}^{\infty} \frac{w^n}{n(1-w^n)} + \frac{\tilde{w}^n}{n(1-\tilde{w}^n)}}
\end{aligned}$$

where

$$f(w) = \prod_{n=1}^{\infty}(1 - w^n) \qquad g(w) = \frac{1}{\sqrt{w}} g_3(w)^{d-1} - \frac{1}{\sqrt{w}} g_4(w)^{d-1} + g_2(w)^{d-1}$$

$$g_3(w) = \prod_{r=\frac{1}{2}}^{\infty}(1 + w^r) \quad g_4(w) = \prod_{r=\frac{1}{2}}^{\infty}(1 - w^r) \quad g_2(w) = \prod_{r=0}^{\infty}(1 + w^r) . \tag{43}$$

We compute the loop-integrals by saddle point approximation for large N, finding

$$\ln(w) \sim -\frac{\pi}{\sqrt{N}} \sqrt{\frac{d-1}{4} - \frac{q^2}{3}} \tag{44}$$

and similarly for $\tilde{w}$.

Therefore, in the elastic limit, for small $\vec{q}$,

$$\begin{aligned}
F_N(\vec{q}^{\,2}) &\underset{N\to\infty}{\sim} \frac{e^{4\pi\sqrt{N}\sqrt{\frac{d-1}{4} - \frac{\vec{q}^{\,2}}{3}}}}{G_c(N)} \pi^d \left(\frac{(d-1)}{4} - \frac{\vec{q}^{\,2}}{3}\right)^{\frac{d}{2}} N^{-\frac{d+2}{2} - \vec{q}^{\,2}}\\
&\underset{\substack{\vec{q}^{\,2}\to 0 \\ N\to\infty}}{\sim} e^{-\frac{4\pi}{3}\sqrt{\frac{N}{d-1}}\vec{q}^{\,2}}
\end{aligned} \tag{45}$$

where in the last line we have simplified the result with

$$G_c(N) \sim e^{2\pi\sqrt{N(d-1)}} \pi^d \left(\frac{d-1}{4}\right)^{\frac{d}{2}} N^{-\frac{d+2}{2}}. \tag{46}$$

Finally, the mass distribution is

$$\mu_N(\vec{r}) = \frac{1}{(2\pi)^d} \int d^d q\, e^{i\vec{q}\cdot\vec{x}} F_N(\vec{q}^{\,2}) = \left(\frac{3}{16\pi^2}\sqrt{\frac{d-1}{N}}\right)^{\frac{d}{2}} e^{-\frac{3}{16\pi}\sqrt{\frac{d-1}{N}}\vec{r}^2}, \tag{47}$$

where $\vec{r}^2 = \vec{x}^2$.

---

[32]Note that [19] sets $\alpha' = 2$, we use $\alpha' = 4$

The bosonic string.

The case of the bosonic string follows the same steps. A few things are different:

- the vertex operator for the probe now is

$$V_{\text{open}}(k, y) = \frac{1}{\sqrt{2\alpha'}} e^{ik \cdot X(y)} \left(iy\xi \cdot \partial_y X(y) \right) \qquad k^2 = k \cdot \xi = 0 \qquad (48)$$

- due to the absence of fermionic excitations, the number of closed string states at fixed large mass squared N is

$$G_c(N) \sim e^{4\pi\sqrt{N}} \sqrt{\frac{d-1}{6}} \pi^d \left(\frac{(d-1)}{6} \right)^{\frac{d}{2}} N^{-\frac{d+2}{2}}; \qquad (49)$$

- the integral (37) becomes now

$$A_o^{\text{zero modes}} = -\int d\epsilon \, \epsilon^{q^2 - 2} e^{-\epsilon(2k \cdot p + 1)} . \qquad (50)$$

Namely, we see that the integral would be divergent also for $q^2 = 1$, corresponding to the exchange of a tachyon. But we are considering the limit $q^2 \to 0$, picking out the graviton pole, so that

$$A_o^{\text{zero modes}} \underset{q^2 \to 0}{\sim} \frac{(2\sqrt{N}E)^2}{q^2} \sqrt{F_b(q^2, E)} \, (2\sqrt{N})^{-q^2} \qquad (51)$$

as for the superstring. Therefore we obtain

- form factor

$$F_N(\vec{q}^{\,2}) \underset{\frac{\vec{q}^{\,2}}{N} \to 0}{\sim} e^{-2\pi\sqrt{\frac{2N}{3(d-1)}}\vec{q}^{\,2}} , \qquad (52)$$

- mass distribution

$$\mu_N(\vec{r}) = \left(\frac{1}{8\pi^2} \sqrt{\frac{3(d-1)}{2N}} \right)^{\frac{d}{2}} e^{-\frac{1}{8\pi}\sqrt{\frac{3(d-1)}{2N}}\vec{r}^{\,2}} . \qquad (53)$$

The number of extended spatial dimensions now can go up to $d = 25$, not only up to 9 as for the superstring.

5.1.2 Corrections

We show here how the lowest terms in the OPE for $y \to 1$ in (35), indeed dominate the amplitude (33) and the result is safe against possible corrections in the considered kinetic and mass range. In particular, within this range, formula (27) appears correct and allows the definition of form factors.

The superstring.

Without any approximations, the amplitude (33) is given by

$$A_o(s, q^2; 1, g_o) \sim \sum_{s=2}^{4} \frac{g_o^2}{G_o(N)} \int d\epsilon \frac{1}{2\pi i} \oint \frac{dw}{w^{N+1}} \frac{g(w)}{f(w)^{d-1}} e^{-\epsilon(2k\cdot p+1)} \psi(\epsilon, w)^{q^2}$$
$$\times \left[-2\partial_\epsilon^2 \ln(\psi(\epsilon, w)) + \chi_s(\epsilon, w) \right] \quad (54)$$

with

$$\psi(\epsilon, w) = (1 - e^{-\epsilon}) \prod_{n=1}^{\infty} e^{-q^2 \frac{w^n}{n(1-w^n)}(e^{n\epsilon} + e^{-n\epsilon})} \quad (55)$$

$$\partial_\epsilon^2 \ln(\psi(\epsilon, w)) = \sum_{n=1}^{\infty} n e^{-\epsilon n} + \sum_{n=1}^{\infty} \frac{n w^n}{(1 - w^n)}(e^{n\epsilon} + e^{-n\epsilon}) \quad (56)$$

$$\chi_s(\epsilon, w) = 2q^2 \frac{\theta_s(\epsilon)^2 \theta_1'(0)^2}{\theta_1(\epsilon)^2 \theta_s(0)^2}, \quad (57)$$

where we have written $y = e^{-\epsilon}$. Note that $\theta_s(z) \equiv \theta_s(\frac{z}{2\pi i}, \frac{\ln(w)}{2\pi i})$ in the usual notation, where the θ_s's are the Theta functions.

Expand for $\epsilon \to 0$:

$$I_\epsilon \sim \int d\epsilon e^{-\epsilon(2k\cdot p+1)} \epsilon^{q^2-2} e^{-2q^2 \sum_n \frac{w^n}{n(1-w^n)}} \left(1 - q^2 + O\left(\frac{w\epsilon^2}{(1-w)}\right)\right)$$
$$\sim (2k \cdot p)^{-q^2+2} \Gamma(q^2) \left(1 + O(\frac{\sqrt{N}}{(k \cdot p)^2})\right). \quad (58)$$

In the limit $-k \cdot p = E\sqrt{N} \gg 1$ that we have been considering (E probe energy, $\sqrt{N}$ tree-level mass for the massive state) our results appear to be valid.

The bosonic string.

The bosonic string case is similar to the superstring one: it can be quickly obtained eliminating from the formulas above the term $\chi(\epsilon, w)$ and substi-

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tuting 1 to $g(w)$, which leads to the result

$$I_\epsilon \sim (2k \cdot p)^{-q^2+2} \Gamma(q^2 - 1) \left(1 + O(\frac{\sqrt{N}}{(k \cdot p)^2}) \right). \tag{59}$$

showing again the validity of our expansion for $E\sqrt{N} \gg 1$.

5.1.3 Identifying the string size operator and recovering the partition function of string states of a given mass and size

In this section we finally show that the operator $\hat{R}$ in (30) does indeed recover the classical value and form given by (14) in the (semi)classical limit.

Note that we have never written or supposed and expression for $\widehat{R^2}$ in term of string oscillators whatsoever. All we have used (and will consistently use here as well) is a procedural definition of the quantum operator through scattering. This we will compare with the classical formula, which is written in terms of string coordinates.

Let us start with the quantum computation. The average value of the quantum operator defined through scattering in the previous section(s) can be easily computed from

$$\langle \widehat{R^2} \rangle = -2d \, \partial_{q^2} F_N(q^2)|_{q^2=0} . \tag{60}$$

In order to compare this to the classical result (14, 18, 19), it is convenient to adopt a canonical ensemble formalism, rather than a microcanonical[33]. Formally, this can be done simply by not performing the integral $\frac{1}{2\pi i} \oint dw \, w^{-N-1}$

---

[33]Let us comment at this point on advantages and disadvantages of the canonical formalism. The canonical ensemble is more convenient than the microcanonical in analyzing values of operators because it somehow preserves the structure of the operator itself (in this case the sum we see in (19)), whereas the microcanonical ensemble would give us just a number, corresponding basically to the sum already performed. In closed string theory, however, the canonical formalism presents a fundamental problem: closed strings must obey the *level matching condition* which sets

$$L_0|\phi\rangle = \tilde{L}_0|\phi\rangle \tag{61}$$

where L_0 ($\tilde{L}_0$) are the Virasoro operators which implement the mass constraint. This, in our case, reduces to

$$N_L = N_R = N \tag{62}$$

for *every single string state*, where N_R depends on tilded oscillators and N_L on not-tilded ones. It appears evident that the naive canonical ensemble cannot ensure that (61) is satisfied for every state and therefore it is not guaranteed that we are tracing only over physical states.

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in the formulas of the preceding section and setting $w = \tilde{w} = e^{-\beta}$, so that

$$Z_{\text{micro}} = tr[\delta(N_R - \hat{N}_R)\,\delta(N_L - \hat{N}_L)] \to Z_{\text{can}} = tr[e^{-\beta\hat{N}_R}\,e^{-\beta\hat{N}_L}] \qquad (63)$$

with β related to the squared mass by $-\partial_\beta \ln(Z_{\text{can}}) = \langle\hat{N}\rangle = N$.

From (60) and the results in the previous section (see (42)), we then find

$$\langle\hat{R}^2\rangle = 4d\sum_{n=1}^{\infty}\left(\frac{w^n}{n(1-w^n)} + \frac{\tilde{w}^n}{n(1-\tilde{w}^n)}\right). \qquad (64)$$

Here we have chosen to write $w, \tilde{w}$ instead of their common value $e^{-\beta}$ for an easier comparison with previous formulas in the paper.

Let us now turn to the classical value given by (18, 19). This has to be compared with the quantum result in the (semi)classical limit. For a discussion about the definition of the classical limit of quantum mechanics see [20]. In the simplest formulation average values of quantum operators are put in relation with classical values. This means that in the limit we can write[34]

$$\alpha^i_{-m}\alpha^i_m \leftrightarrow \langle\hat{\alpha}^i_{-m}\hat{\alpha}^i_m\rangle. \qquad (65)$$

Now, neglecting the motion of the center of mass, states of a closed (therefore periodic) string are standing waves of various integer frequencies n. We will use a semiclassical limit, instead than a fully classical one, in order to preserve the discreteness of energies of standing waves. The true classical behavior can then be obtained at high temperature (small β), large quantum numbers.

The total energy of the string is

$$N = N_L + N_R \qquad (66)$$

$$N_L = \sum_{n\geq 1,i}\alpha^i_{-n}\alpha^i_n \equiv \sum_{n\geq 1,i}nN^i_{L,n} \qquad (67)$$

$$N_R = \sum_{n\geq 1,i}^{+\infty}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n \equiv \sum_{n\geq 1,i}nN^i_{R,n} \qquad (68)$$

As is well known, then, the occupation number relative to a single wave energy level n in the canonical ensemble is given by $N^i_n = (e^{\beta n} - 1)^{-1}$. By substituting this in (18, 19), always in the semiclassical limit (65), we obtain

$$\mathcal{R}^2_{\text{cl}} = 4d\sum_{n=1}^{\infty}\left(\frac{w^n}{n(1-w^n)} + \frac{\tilde{w}^n}{n(1-\tilde{w}^n)}\right). \qquad (69)$$

---

[34]Where not explicitly written, no sum over the index i is performed in the following.

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We clearly see that in this limit

$$\langle \widehat{R^2} \rangle = \mathcal{R}_{\text{cl}}^2 \,. \tag{70}$$

The quantum operator will actually also have a zero point contribution from the normal ordering. This latter gives origin to the sub-leading factor N^{-q^2} in (42) which is negligible for $N \to \infty$.

As we already stressed, the computations of the quantum average (64) and of the classical value (18, 69) are completely independent: the former evaluates a quantum operator defined in a consistent procedural way through scattering with no reference to an expression in terms of string oscillators, the latter uses a classically well defined formula in terms of oscillators in the semiclassical limit.

Having shown that the average values of $\widehat{R}^2$ correctly gives the classical value (14) of the string size in the semiclassical limit, we can identify (30) with the the partition function $G_c(N, R)$ for string states with mass level N and size R and write

$$G^* = G_c(N, R) \,. \tag{71}$$

5.1.4 Partition function for strings at fixed mass and size

Using now formulas (28, 29, 30), we obtain

- for the **superstring**

  - the partition function for closed string states with fixed R, N

  $$G_c(N, R) = \frac{2}{\Gamma(\frac{d}{2})} \left(\frac{3\sqrt{d-1}}{16\pi^2 \sqrt{N}} \right)^{\frac{d}{2}} \left(\frac{R}{\sqrt{N}} \right)^{d-1} \frac{e^{\pi\sqrt{d-1}\left(2\sqrt{N} - \frac{3}{16\pi^2 \sqrt{N}} R^2 \right)}}{N^{\frac{3}{2}}} \tag{72}$$

  - and the entropy

  $$\begin{aligned} S &= \ln(G_c(\sqrt{N}, R)) \\ &\sim 2\pi\sqrt{N}\sqrt{d-1} - \frac{3\sqrt{d-1}}{16\sqrt{N}\pi} R^2 + \ln\left(\frac{R^{d-1}}{\sqrt{N}^{\frac{3}{4}d+1}} \right) \end{aligned} \tag{73}$$

- for the **bosonic string**

  - the partition function for closed string states with fixed R, N

  $$G_c(N, R) = \frac{2}{\Gamma(\frac{d}{2})} \left(\frac{\sqrt{3(d-1)}}{8\sqrt{2}\pi^2 \sqrt{N}} \right)^{\frac{d}{2}} \left(\frac{R}{\sqrt{N}} \right)^{d-1} \frac{e^{\pi\sqrt{d-1}\left(\sqrt{\frac{8N}{3}} - \frac{\sqrt{3}}{8\sqrt{2}\pi^2 \sqrt{N}} R^2 \right)}}{N^{\frac{3}{2}}} \tag{74}$$

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– and the entropy

$$
\begin{aligned}
S &= \ln(G_c(N, R)) \\
&\sim 4\pi\sqrt{N}\sqrt{\frac{d-1}{6} - \frac{\sqrt{3\,(d-1)}}{8\sqrt{2}\,\sqrt{N}\,\pi}R^2} + \ln\left(\frac{R^{d-1}}{\sqrt{N}^{\frac{3}{4}d+1}}\right).
\end{aligned} \tag{75}
$$

By maximizing the entropy with respect to R at fixed N, we find that the majority of string states have the following favored values of the size:

- for the superstring:

$$
R^2_{\text{max number}} = \frac{8\pi\,\sqrt{d-1}}{3}\sqrt{N} \tag{76}
$$

- for the bosonic string

$$
R^2_{\text{max number}} = \frac{4\sqrt{2}}{3}\,\pi\,\sqrt{d-1}\sqrt{N}, \tag{77}
$$

The average radius is instead:

- for the superstring:

$$
R^2_{\text{average}} = \frac{8\pi\,d}{3}\sqrt{\frac{N}{d-1}} \tag{78}
$$

- for the bosonic string

$$
R^2_{\text{average}} = 4\sqrt{2}\,\pi\,d\sqrt{\frac{N}{3\,(d-1)}}, \tag{79}
$$

which are in agreement with results (obtained with various approximations) in the literature.

A few remarks are useful at this point. First, we can appreciate in our derivation, the importance of the factorization property of (string) amplitudes: as we already mentioned, factorizing two external legs of an amplitude, the momentum square q^2 flowing along the connecting propagator is a continuous variable, allowing analytic continuation. Therefore (22) is computable, using formula (30), in a perfectly consistent way within string theory[35], although the presence of an off-shell insertion $\delta(R^2 - \widehat{R^2}_s)$. The realization of

_____

[35]Starting form the well-defined on-shell amplitude (23).

this point, is the key technical achievement that allows the computation of the quantum partition function.

We could also wonder whether our result depends on the ordering of the two delta insertions in (22), since we do not expect $\widehat{R^2}_s$ and $\hat{N}_{R(L)}$ to commute. In any case, we can be reassured by the fact that obviously $\delta(R^2 - \widehat{R^2}_s) \, \delta(N - \hat{N}_{R(L)})$ and $\delta(N - \hat{N}_{R(L)}) \, \delta(R^2 - \widehat{R^2}_s)$ yield the same result when traced over, and, furthermore, we are working with very massive string states, for which it is also reasonable to take a semi-classical limit.

As a final remark, we can also see that the quantum computation we have performed has clarified the link with the random walk approximation that has been used in the past. Indeed, in the random walk picture a string of mass M is described as a Gaussian of width proportional to $\sqrt{M}$. The mass distributions we have obtained in formulas (47, 53) for the bosonic string and the superstring have precisely that form. This compares also to the discussion in [5], section 2.

5.2   States carrying Neveu-Schwarz charges

In this section we consider states carrying non-zero Neveu-Schwarz type charges. The implementation of the relative delta functions in (12, 22) is easily achieved by fixing the Kaluza-Klein and winding mode numbers for the states in the ensemble. We therefore report here the notation and the results.

5.2.1   Non-BPS states

The results obtained in the previous sections can be extended to ensembles of string states carrying Neveu-Schwarz charges Q_R, Q_L. We have to distinguish states according to their mass and their winding and Kaluza-Klein mode numbers (m^i, n^i), such that:

$$Q^i_{L,R} = \left(\frac{n^i}{r^i} \pm \frac{m^i r^i}{4} \right) \tag{80}$$

$$Q^2_{L,R} = \sum_i Q^{i\,2}_{L,R}, \tag{81}$$

where r^i is the radius[36] of compactification in the i-th compactified direction.

The mass-shell condition and the Virasoro constraint $(L_0 - \tilde{L}_0)|\phi\rangle = 0$

---

[36]Recall that we set $\alpha' = 4$ and express everything in units of α'.

read

$$M^2 = Q_L^2 + N_L \tag{82}$$
$$= Q_R^2 + N_R \tag{83}$$
$$N_L - N_R = -\sum_i n^i m^i. \tag{84}$$

where L, R indicate respectively the left- and right-moving sectors.

We define our microcanonical system by fixing charge and squared mass, which implies fixing the values N_L, N_R of the operators $\hat{N}_L$, $\hat{N}_R$. We consider large N_L, N_R.

Then, defining

$$\mathcal{N} = \sqrt{N_L} + \sqrt{N_R}, \tag{85}$$

• for the **superstring**

  – the partition function for closed string states with fixed size, mass, charge is

  $$G_c \sim \frac{2}{\Gamma(\frac{d}{2})} \left(\frac{3\sqrt{d-1}}{8\pi^2 \mathcal{N}} \right)^{\frac{d}{2}} \left(\frac{R}{N_L^{\frac{1}{4}} N_R^{\frac{1}{4}}} \right)^{d-1} \frac{e^{\pi\sqrt{d-1}\left(\mathcal{N} - \frac{3}{8\pi^2 \mathcal{N}} R^2\right)}}{N_L^{\frac{3}{4}} N_R^{\frac{3}{4}}} \tag{86}$$

  – and the entropy

  $$S = \ln(G_c)$$
  $$\sim \pi \mathcal{N} \sqrt{d-1} - \frac{3\sqrt{d-1}}{8\mathcal{N}\pi} R^2 + \ln\left(\frac{R^{d-1}}{N_L^{\frac{d+2}{4}} N_R^{\frac{d+2}{4}} \mathcal{N}^{\frac{d}{2}}} \right) \tag{87}$$

  with

  $$R^2_{\text{max number}} = \frac{4\pi\sqrt{d-1}}{3} \mathcal{N} \tag{88}$$
  $$R^2_{\text{average}} = \frac{4\pi d}{3} \frac{\mathcal{N}}{\sqrt{d-1}} \tag{89}$$

• for the **bosonic string**

  – the partition function for closed string states with fixed size, mass, charge is

  $$G_c = \frac{2}{\Gamma(\frac{d}{2})} \left(\frac{\sqrt{3(d-1)}}{4\sqrt{2}\pi^2 \mathcal{N}} \right)^{\frac{d}{2}} \left(\frac{R}{N_L^{\frac{1}{4}} N_R^{\frac{1}{4}}} \right)^{d-1} \frac{e^{\pi\sqrt{d-1}\left(\sqrt{\frac{2}{3}}\mathcal{N} - \frac{\sqrt{3}}{4\sqrt{2}\pi^2 \mathcal{N}} R^2\right)}}{N_L^{\frac{3}{4}} N_R^{\frac{3}{4}}} \tag{90}$$

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– and the entropy

$$
\begin{aligned}
S &= \ln(G_c) \\
&\sim 2\pi\mathcal{N}\sqrt{\frac{d-1}{6}} - \frac{\sqrt{3(d-1)}}{4\sqrt{2}\,\mathcal{N}\,\pi}R^2 + \ln\left(\frac{R^{d-1}}{N_L^{\frac{d+2}{4}} N_R^{\frac{d+2}{4}} \mathcal{N}^{\frac{d}{2}}}\right) .
\end{aligned}
\tag{91}
$$

5.2.2 BPS states

We study, now, BPS configurations of fundamental superstrings. They are states with:

$$
M^2 = Q_L^2, \quad N_L = 0, \quad N_R = \sum_i n^i m^i .
\tag{92}
$$

We find:

- the partition function for closed BPS string states with fixed size, mass, charge

$$
G_c = \frac{2}{\Gamma(\frac{d}{2})}\left(\frac{3\sqrt{d-1}}{8\pi^2\sqrt{N_R}}\right)^{\frac{d}{2}}\left(\frac{R}{N_R^{\frac{1}{4}}}\right)^{d-1}\frac{e^{\pi\sqrt{d-1}\left(\sqrt{N_R}-\frac{3}{8\pi^2\sqrt{N_R}}R^2\right)}}{N_R^{\frac{3}{4}}} .
\tag{93}
$$

- and the entropy

$$
\begin{aligned}
S &= \ln(G_c) \\
&\sim \pi\sqrt{N_R}\sqrt{d-1} - \frac{3\sqrt{d-1}}{8\sqrt{N_R}\pi}R^2 + \ln\left(\frac{R^{d-1}}{N_R^{\frac{d+1}{2}}}\right) .
\end{aligned}
\tag{94}
$$

It is interesting to note that the average radius for this ensemble is

$$
R^2_{\text{average}} = \frac{4\pi\,d}{3}\sqrt{\frac{N_R}{d-1}} .
\tag{95}
$$

which is larger than the Schwarzshild radius at transition/matching point (see section 6.1).

6 Size distribution for highly excited string states with self-interactions

The counting of states at a given mass level is affected by the self-interaction of the string, unless we are considering supersymmetric configurations, which enjoy a protection mechanism for the mass.

The idea is that the formulas for the entropy obtained in sections 5.1 and 5.2 will receive corrections due to renormalization of the mass and size, such that the partition function will be dominated by string states with a typical size within the Schwarzschild radius of the correspondent black hole at the transition/matching point.

The questions we want to address are:

- what is the distribution of string states at a certain mass and charge in terms of the size at non-zero coupling? Are small sizes preferred?

- how does the value of the Schwarzschild radius emerges from the string point of view[37]?

- what is the minimal size a string state can attain at non-zero coupling?

In order to answer those questions we need to compute the effect of interactions on string states. These will modify the dynamical equations of the states and their partition function. We will study self-energy (mass-renormalization) corrections for string states elaborating on [4]. For simplicity, we consider $|Q_L^i| = |Q_R^i|$, $N_L = N_R$ and define $Q^i \equiv Q_L^i$.

Formally, in operatorial form, the average squared mass-shift for states constrained in both mass, charges and size would be obtained from the formula

$$\overline{\Delta M^2}|_{N,Q,R} = G_c(N,Q,R)^{-1} \tag{96}$$
$$\times \text{tr}[\widehat{\Delta M}^2 \, \delta(N - \hat{N}_R) \, \delta(N - \hat{N}_L) \delta(Q_R - \hat{Q}_R) \delta(Q_L - \hat{Q}_L) \delta(R - \hat{R})]$$

where $\widehat{\Delta M}^2$ is an operator yielding the squared mass shift once applied to a set of states[38].

Integrating over R and dividing by $G_c(N,Q)^{-1}$, formula (96) translates into

$$\overline{\Delta M^2}|_{N,Q} = \int dR \, \overline{\Delta M^2}|_{N,Q,R} \, \rho_c(N,Q,R). \tag{97}$$

where $\rho_c(N,Q,R)$ is the *density* of string states at given mass-level, charge and size, equal to

$$\rho_c(N,Q,R) \equiv \frac{G_c(N,Q,R)}{G_c(N,Q)} \,. \tag{98}$$

---

[37]We know it is important because we are following *two* descriptions of the same system, according to the correspondence principle: the classical black hole and the quantum string ensemble.The Schwarzschild radius is characteristic of the classical description; does it arise as a special quantity also from the point of view of self-interacting strings? And how?

[38]It could be obtained opportunely normalizing the real part of the one-loop S-matrix operator.

and $\overline{\Delta M^2}|_{N,Q}$ is the average squared mass-shift at fixed N, Q[39].

In [4] it was obtained[40]

$$\overline{\Delta M^2}|_{N,Q} = -g_s^2 (M_0^2 - Q^2)^{1 + \frac{3-D}{4}}, \tag{99}$$

where we have used the definition of the tree-level mass

$$M_0^2 = N + Q^2. \tag{100}$$

It is possible to see that the squared mass-shift becomes non-negligible (of order one) for

$$g_{se} \sim (M_0^2 - Q^2)^{\frac{d-6}{8}}, \tag{101}$$

in analogy with the expectations form the field theory argument in formula 3.3 in [5] (they consider only $Q^2 = 0$).

In order to solve equation (97), it is useful to propose an ansatz. As it was discussed in section 5 of [4], the formula (99) is constituted by two factors with different origins. In particular, the factor $(M_0^2 - Q^2)^{\frac{3-D}{4}}$ is related to the spatial range of the interaction which was found to provide the dominant contribution to the mass-shift (namely gravitational interactions). In [4], it was also discussed how this was in fact given by the average length of the massive string (we have indeed found $R^2_{\text{average}} \sim \sqrt{M_0^2 - Q^2}$ in sections 5.1.4, 5.2.1).

We therefore consider the ansatz

$$\overline{\Delta M^2}|_{N,Q,R} = -g_s^2 c (M_0^2 - Q^2) R^\alpha, \tag{102}$$

with c a suitable proportionality constant.

From (97, 98, 99) and the results for $G_c(N, Q, R)$ obtained in sections 5.1.4, 5.2.1, we determine the correct power α

$$\alpha = 2 - d, \tag{103}$$

so that[41]

$$\overline{\Delta M^2}|_{M,Q,R} = -g_s^2 c (M^2 - Q^2) R^{2-d}, \tag{105}$$

---

[39] $\overline{\Delta M^2}|_{N,Q} = G_c(N,Q)^{-1} \operatorname{tr}[\widehat{\Delta M}^2 \delta(N - \hat{N}) \delta(Q_R - \hat{Q}_R) \delta(Q_L - \hat{Q}_L)]$, $Q^i = Q_L^i$.

[40] In [4] g_s was redefined to get rid of a positive constant of order one in the result of the average squared mass shift. However, we can still retain the formula $G_N \sim g_s^2 (\alpha')^{\frac{d-1}{2}}$, relating Newton's constant to the string coupling for small curvature radii $r_i \sim \sqrt{\alpha'}$ of compactified dimensions, since at this level we do not pay attention to constant factors of order one. This implies that we will not be able to account for the specific proportionality factor of $\frac{1}{4}$ in Beckenstein's formula for the entropy.

[41] In this formula we write the mass-shift in terms of the true mass, which is probably even a more accurate estimate. Note also that this result for the correction is valid in

with $c = \frac{\Gamma(\frac{d}{2})}{\pi}\left(\frac{16}{3\sqrt{d-1}}\right)^{\frac{d-2}{2}}$. In the following we set $c = 1$. There is no loss of generality in doing this because we can opportunely redefine g_s. Indeed, we are not going to pay attention to factors of order one.

Note that our result is different from that in [6] in the power of the mass (they consider only $Q^2 = 0$). As we said in section 2, around (4), their result would imply that we cannot apply perturbation theory uniformly on the whole string spectrum in lower dimensions, certainly not in the limit of large masses. With the result (99), obtained from well-defined string amplitudes in [4], perturbation theory is generally viable on the string spectrum[42].

We can now discuss how the string distribution in terms of mass, charge and size is modified by the corrections. The partition function modifications proceed from[43]

$$G_c(M_0, Q, R_0) = G_c(M^2, Q, R), \tag{106}$$

where we neglect renormalization of R and use

$$M_0^2 = M^2 - \overline{\Delta M^2}_{|M,Q,R} \tag{107}$$

from the definition of mass-shift. This translates into[44]

$$M_0^2 - Q^2 = M^2 - Q^2 + g_s^2(M^2 - Q^2)R^{2-d}. \tag{108}$$

The important effects in $G_c(M^2, Q, R)$ arise from the exponential factor (see (86)), which now at leading order[45] is

$$e^{2\pi\sqrt{d-1}\left(\sqrt{M^2-Q^2}+g_s^2\frac{\sqrt{M^2-Q^2}}{2R^{d-2}}-\frac{3}{32\pi^2\sqrt{M^2-Q^2}}R^2\right)} \tag{109}$$

---

perturbation theory only for sizes larger than a certain minimal value. Alternatively, it is possible to find solutions to the equation (97) that deviates from (105) at small R and are valid for all sizes, such as

$$\overline{\Delta M^2}_{|M,Q,R} = -g_s^2 c\frac{M^2 - Q^2}{R^{d-2}}\frac{2^{\frac{d-2}{2}}}{\Gamma(\frac{d}{2})}\left(1 - \frac{\Gamma(\frac{d}{2}, 3\sqrt{d-1}\,R^2\,(16\pi\sqrt{N})^{-1})}{\Gamma(\frac{d}{2})}\right), \tag{104}$$

where c is the same as in (105) and $\Gamma(a, x)$ is the incomplete gamma function.

We prefer (105) because it is conceivable that perturbations theory breaks down for very small sizes. However, the results we find remain in general true even using (104).

[42]This does not exclude that particular sets of states, not representing significant portion of the string spectrum (sets of "measure zero") and therefore not affecting the average, could have larger corrections and therefore not be suited for a perturbative treatment.

[43]We consider adiabatic variations of the coupling, for which the entropy is unchanged, so that $\log(G_c(M_0, Q, R)) = \log(G_c(M^2, Q, R))$.

[44]The winding number and also the Kaluza-Klein mode (T-dual to it) are not renormalized.

[45]We neglect renormalization of R_0, that is we reckon that $\overline{\Delta R^2}_{|M,Q,R} \ll \overline{\Delta M^2}_{|M,Q,R}$.

We can see from this that the behavior of the string partition function and the entropy (its logarithm) changes for

$$g_s^2 \frac{\sqrt{M^2 - Q^2}}{2R^{d-2}} \geq 1 \tag{110}$$

when it becomes dominated by strings of size $R \lesssim R_b$, where

$$R_b \sim (g_s^2 \sqrt{M^2 - Q^2})^{\frac{1}{d-2}} . \tag{111}$$

But R_b is indeed the value of the Schwarzschild radius for a charged black hole in d spatial dimensions[46]. In the description of our string distribution, it becomes important when $R_b \geq l_s$, where l_s is the string length, which occurs at $g_s \sim (M^2 - Q^2)^{-\frac{1}{4}}$, in accordance with the correspondence principle[47] in [5]. The value $R^2 \sim \sqrt{M^2 - Q^2}$ remains, in any case, at small g_s, a local maximum of the partition function/entropy.

Another interesting question that we can try to ask ourselves at this point regards the minimal value for the size of a string state with self-interaction. We can estimate it from the modification to the equation of motion: for a state $|\phi\rangle$ of true squared mass M^2 the energy is

$$E_\phi^2 = \vec{p}^2 + M^2 = \vec{p}^2 + N - g_s^2 \frac{M^2 - Q^2}{R^{d-2}} . \tag{112}$$

We can make a rough estimate for the minimal radius (and this suffices for our work here), as usual in quantum mechanics, through the Heisenberg uncertainty principle. That is, we roughly expect[48] $p^i \sim \frac{1}{R}$ and then we estimate the energy of the minimum E_0 by minimizing (112) with respect to R.

---

[46] We consider here black holes obtained by the usual procedure of lifting a $D = d + 1$ dimensional Schwarzschild solution to $D + 1$ dimensions, boosting along the new extra dimension and reducing down to D dimensions again (see [21]). R_b is the value of the horizon radius r_E in the Einstein frame when M is identified with the ADM mass M_{BH} (see [5], for $d = 3$, their result can be extended to $d > 3$. Consider that we have $|Q_L^i| = |Q_R^i|$). We define the horizon radius in the Einstein frame as $r_E^2 = e^{-\frac{4}{D-2}\phi} r_S^2$ where ϕ is the dilaton and r_S is the horizon radius in the string frame. In this way the area of the black hole horizon is $A \sim r_E^{D-2}$. Of course, in view of the correspondence principle, the identification $M_{BH} = M$ is done at a specific value of the coupling. In particular, the matching in this case should not be done when $r_E \sim l_s$, but when $r_S \sim l_s$ (see [5]).

[47]This is strictly true when $Q^2 = 0$: as we said in footnote 46, when charges are present, the match occurs at $r_S \sim l_s$. However, if we trust the classical description further up when $r_E \sim l_s$, which is sensible when $N = M_o^2 - Q^2 \gg 1$, then $G_N \sim \sqrt{M^2 - Q^2}$ there.

[48]In our units $\hbar = 1$.

We immediately see the difference between cases $d = 3$, $d = 4$, $d \geq 5$. For $d = 3$ we find a lowest state for

$$R = \frac{2}{g_s^2(M^2 - Q^2)} \tag{113}$$

for $d \geq 5$ instead there is no lowest state, and apparently the energy has no lower bound. This would signal an instability. But it actually occurs outside the domain of validity of perturbations theory, since it would show up when $N - g_s^2 \frac{M_0^2 - Q^2}{R^{d-2}} < 0$ and therefore when $\overline{\Delta M^2}|_{M,Q,R} > M_0^2$. This means that in this case we cannot trust our first order perturbative corrections to give us an exhaustive insight on what happens for extremely compact string states. For $d = 4$ the energy is negative past a critical coupling.

6.1   Matching point for BPS states

The horizon radius (and therefore the entropy) for the BPS black holes corresponding to the states considered here vanishes. The string entropy, instead, as shown in section 5.2.2, is non-zero and grows with the mass. It would seem that the string/black hole correspondence principle does not work in this case.

Sen's proposal [22] at this point was that the entropy formula to be considered should involve the area of the surface at the "stretched horizon" and not the Schwarzschild one. The rationale behind this is that for BPS black holes the curvature of the classical geometry is comparable to the string scale already at the stretched horizon and therefore the classical description is unreliable already there, where string effects show up. Following this line of reasoning, in a class of solutions it was shown that a higher derivative corrected computation of entropy matches the string entropy [23].

We know that the counting of BPS states, and therefore their entropy, for fixed mass and charge do not receive corrections due to the vanishing of their two-points torus amplitude. Nevertheless, it would be interesting to study the configuration of the string, since the matching is at non-zero string coupling.

What is the nature of the corrections that would favor more compact states is not obvious. In [10] it was conjectured a correction whose form was very similar to the one obtained from one-loop self-energy corrections; however, it was not clarified which kind of string diagram would produce it.

7 Discussion and Conclusions

The main new results of this paper are twofold: 1) the computation of the partition function for closed very massive free string states in the microcanonical ensemble at given (large tree-level) mass, charges (Neveu-Schwarz) and size, 2) the study of the dynamics of the string in presence of self-interactions and the resulting modifications of the partition function at given (large true) mass, charges and size.

At tree-level, our quantum computation shows how the (expected) random walk picture for the string arises. On the other hand, the self-interaction of the string modifies the distribution of the string states in an important way[49]. These results allows us to clarify the correspondence ([2, 3]) between string states an black holes in non-supersymmetric configurations.

In particular, our results make many of the conclusions in [5] (we think) physically clearer by directly computing the relevant quantities for a well-defined string ensemble. The computations, indeed, are performed rigorously in string theory formalism in the asymptotic limit of large string masses (highly excited states). In comparing our results with those in [5], note however that we perform our analysis using a microcanonical ensemble of closed strings and not a canonical one for open strings as in [5].

When our results differ from [5], they make even more compelling the existence of a correspondence (complementarity) between strings and black holes. For example, [5] seems to find an instability for $d = 5$ and, from the string $\rightarrow$ black hole side of the correspondence, the string seems to collapse to a black hole at a value for the coupling lower than the critical one expected from the correspondence principle (which was found investigating the black hole $\rightarrow$ string side of the correspondence). Also, [5] suggested that in $d > 6$ most excited string states would never correspond to a black hole, at any value of the coupling.

The physical picture that emerges form our computations, shows instead that at sufficiently large coupling g_s, in any dimensions the string ensemble will be dominated by typical strings of size[50] $R \lesssim (g_s^2 M)^{\frac{1}{d-2}}$ which corresponds to the black hole Schwarzschild radius and which is of the order of the string scale precisely at the expected correspondence point $g_s \sim M^{-\frac{1}{2}}$. We find that for $d > 6$, this occurs when the average self-gravity correction is not yet strong (see (99, 101)). The details are different between dimensions $d = 3$, $d = 4$ and $d \geq 5$.

---

[49]We will not discuss here the details involved in computing and analyzing (99), but we direct the reader to [4].

[50]To simplify the notation here we take $Q^2 = 0$.

Let us focus on the best understood case ($d = 3$). The entropy of the self-gravitating string is dominated by the lowest bound state and is given by

$$S(M, Q) \sim 2\pi\sqrt{d-1}\sqrt{M^2 - Q^2}\left(1 + \frac{g_s^4(M^2 - Q^2)}{4}\right). \qquad (114)$$

Let us consider now for simplicity the case $Q^2 = 0$ and obtain the temperature of ensemble of closed string by differentiating with respect to M. We find:

$$T \sim T_H\left(1 - \frac{3}{4}g_s^4 M^2\right), \qquad T_H = \frac{1}{2\pi\sqrt{d-1}} \qquad (115)$$

which shows the diminishing of the Hagedorn temperature[51] T_H in agreement with [5]. Note however that we are not describing our system in a canonical ensemble and therefore the discussion of phase transitions is different from the one in canonical formalism. We will not deal with these interesting questions here, but leave them for future investigation.

8 Acknowledgments

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